TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 227, 1977

STRUCTURE OF SUBALGEBRAS BETWEEN L^∞ AND H^∞

BY

SUN-YUNG A. CHANG(1)

ABSTRACT. Let B be a closed subalgebra of L^∞ of the unit circle which contains H^∞ properly. Let C_B be the C^*-algebra generated by the inner functions that are invertible in B. It is shown that the linear span $H^\infty + C_B$ is equal to B. Also, a closed subspace (called VMO_B) of BMO (space of functions of bounded mean oscillation) is identified to which B bears the same relation as L^∞ does to BMO.

1. **Introduction.** Let L^∞ denote the space of essentially bounded Lebesgue measurable functions on the unit circle. Let H^∞ denote the subspace of functions in L^∞ with vanishing negative Fourier coefficients. The algebras studied here are the closed subalgebras of L^∞ which contain H^∞ properly.

The interest in such algebras originated, to a large extent, in a question asked by R. G. Douglas [1]. Suppose B is a closed subalgebra of L^∞ which contains H^∞. Let B_I be the closed subalgebra of L^∞ generated by H^∞ and the complex conjugates of those inner functions (that is, unimodular H^∞ functions) that are invertible in B. Douglas asked: Is B_I always equal to B? (If so, we call B a Douglas algebra.) The question has recently been answered affirmatively by D. Marshall [2] using a result in [3].

Another interesting question arose in the study of Douglas' problem. All subalgebras of L^∞ containing H^∞ which have been investigated in detail share certain structural characteristics. (For a survey of some of this work see [4].) Sarason [5] pointed out the phenomenon and asked, in particular, the following question: Let B be a closed algebra between L^∞ and H^∞. Let C_B be the C^*-algebra generated by the inner functions that are invertible in B (which is the same algebra generated by all quotients of inner functions invertible in B). Is it true in general that $B = H^\infty + C_B$?

The largest such algebra is L^∞. R. G. Douglas and W. Rudin [6] have

Presented to the Society, November 15, 1975 under the title *C^*-algebras contained in subalgebras between L^∞ and H^∞*; received by the editors December 3, 1975.

AMS (MOS) subject classifications (1970). Primary 30A76, 43A40, 46J35.

(1) This work was supported under NSF Grant MPS 75-06675.

proved that L^∞ is the closed algebra generated by quotients of inner functions in H^∞. By a theorem of K. Hoffman and I. M. Singer [7], the smallest such subalgebra is the closed algebra generated by H^∞ and C, where C denotes the algebra of continuous functions on ∂D. D. Sarason [8] has proved that this closed algebra is in fact equal to $H^\infty + C$, the linear hull of H^∞ and C. For E an arbitrary subset of ∂D, let L_E denote the set of functions in L^∞ that are continuous on E. A. Davie, T. Gamelin and J. Garnett [9] have extended the preceding result by showing that the closed subalgebra of L^∞ generated by H^∞ and L_E coincides with $H^\infty + L_E$, the linear hull of H^∞ and L_E. In addition, they have proved that L_E is the algebra generated by functions of the form $b_1\bar{b}_2$, where b_1 and b_2 are Blaschke products whose zeros do not cluster at any point of E.

Sarason's question has been answered affirmatively for some other subalgebras of L^∞ containing H^∞ [10], [11]. In §2 below, I shall give an affirmative answer to this question in general.

In the examples mentioned above, a concrete description of the C^*-algebra C_B in terms of properties of the original algebra B has been given. The effort to get similar results for general C_B has not been successful. However, Sarason [11] has studied a closed subspace VMO (functions of vanishing mean oscillation) of BMO (functions of bounded mean oscillation), and obtained results which, roughly speaking, tell that the relation of the closed algebra $H^\infty + C$ to the space VMO is like the relation of L^∞ to BMO. For example, if we let QC (for quasi-continuous) denote the C^*-algebra $\overline{(H^\infty + C)} \cap (H^\infty + C)$, then $VMO = QC + \widetilde{QC} = C + \tilde{C}$. (For a subspace S of L^∞, we let $\bar{S}$ denote the collection of complex conjugates of functions in S, and $\tilde{S}$ denote the collection of harmonic conjugates of functions in S.) Using a method similar to Sarason's, S. Axler [12], T. Weight [13] and the author [10] have obtained parallel results for other specific subalgebras between H^∞ and L^∞. In §§3 and 4 below, I generalize these results and obtain, for any subalgebra B between L^∞ and H^∞, a corresponding closed subspace VMO_B of BMO. Theorems 6 and 7 say that B bears the same relation to VMO_B as L^∞ does to BMO.

The following are some notations used in this paper. Let L^1 and L^2 denote Lebesgue spaces of integrable and square integrable functions with respect to normalized Lebesgue measure on the unit circle. Let H^1 and H^2 denote the corresponding Hardy spaces. Let H_0^1 and H_0^∞ denote the functions in H^1 and H^∞, respectively, with mean value 0. For a subset S of L^∞, let $H^\infty[S]$ denote the closed subalgebra of L^∞ generated by H^∞ and S. If S consists of a single function f, denote $H^\infty[S]$ by $H^\infty[f]$. For a function f in L^∞, the norm $\|f\|$ is the essential supremum of $|f|$ on ∂D. For a subspace S of L^∞, the distance of f to S is given by $d(f, S) = \inf\{\|f - g\|, g \in S\}$. Finally, for a function f in

L^1 of ∂D, let $f(re^{i\theta})$ denote the harmonic extension of f into the unit disk D by means of Poisson's formula, i.e.,

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)P_r(\theta - t)\,dt,$$

where $P(r,t) = (1 - r^2)/(1 - 2r\cos t + r^2)$ is the Poisson kernel.

I am deeply indebted to Professor D. Sarason, not only for suggesting the problem and giving invaluable aid during the research, but also for teaching me about *BMO* in his course. I am also grateful to Professor J. Garnett for various helpful suggestions and discussions. I would like also to express my thanks to Professor C. Fefferman for a stimulating discussion about the subject.

2. **Structure of subalgebras between L^∞ and H^∞.** We will begin with a theorem of A. Bernard, J. Garnett and D. Marshall.

THEOREM 1 ([14, THEOREM 2.1]). *Let $f \in L^\infty$, $d(f, H^\infty) < 1$. Then there is unimodular function $u_0 \in L^\infty$ such that $u_0 \in f + H^\infty$ and $d(u_0, H_0^\infty) = 1$.*

In [15] there is a sharper theorem: If $f \in L^\infty$, and if the coset $f + H^\infty$ contains two functions of norm $\leqslant 1$, then it contains a unimodular function. But the proof of Theorem 1 in [14] is much shorter, and gives in addition a unimodular function in the coset whose distance to H_0^∞ is 1. We briefly indicate how to find the function here.

PROOF. We may assume without loss of generality that $\|f\| < 1$. Write $a = \sup\{|\int u\,d\theta/2\pi| : u \in f + H^\infty, \|u\| \leqslant 1\}$. Since H^∞ is weak-star closed, a compactness argument shows there exists an extremal function u_0 such that $|\int u_0\,d\theta/2\pi| = a$. It follows easily from the extremal property of u_0 that $d(u_0, H_0^\infty) = 1$ and $\|u_0\|_\infty = 1$. Now an argument using the duality between L^∞/H_0^∞ and H^1 can be applied to show that any function u in L^∞ with the properties $\|u\|_\infty \leqslant 1$, $d(u, H^\infty) < 1$ and $d(u, H_0^\infty) = 1$ is unimodular. In particular, u_0 is unimodular. (Details of the proof are in [14].)

The following fact was indicated to me in a discussion with D. Sarason.

THEOREM 2. *Let f be a unimodular function in L^∞. If $d(f, H^\infty) = 1$, and $d(f, H^\infty + C) < 1$, then $\bar{f} \in H^\infty[f]$.*

PROOF (SARASON). For the given $f \in L^\infty$, let T_f be the Toeplitz operator associated with f, i.e., T_f is the bounded map from H^2 to H^2 defined by $T_\varphi(h) = P(\varphi h)$, where P is the orthogonal projection from L^2 to H^2.

Since f is unimodular and $d(f, H^\infty) = 1$, T_f is not left-invertible ([16, Lemma 1]). Since $d(f, H^\infty + C) < 1$, T_f is left-Fredholm [17]. Thus the range of T_f is a closed subspace of H^2, and the null space of T_f is nontrivial. It then follows from a theorem of Coburn [18] that the null space of T_f^* is trivial.

Hence T_f is onto and so right invertible. Thus, $T_{\bar{f}}$ is Fredholm, which implies that $\bar{f}$ is in $H^\infty[f]$ by [19, 7: 33]. ([19] also lists all other results concerning the Toeplitz operator used here.)

The next lemma follows easily from the above theorems.

LEMMA 1. *Let $h \in H^\infty$, and let b be an inner function such that $d(\bar{b}h, H^\infty) < 1$. Then there is an inner function b_0 in $h + bH^\infty$ with $\overline{b_0} \in H^\infty[\bar{b}]$.*

REMARKS. (1) The existence of inner functions b_0 in $h + bH^\infty$ is proved in Satz 7 of Nevanlinna's work [20] in case b is a Blaschke product. But it is not clear whether the b_0 constructed in his proof is invertible in $H^\infty[\bar{b}]$.

(2) If $b = z^n$, then Lemma 3 in slightly generalized form is contained in Carathéodory's classical theorem [21]: If $h \in H^\infty$, $\|h\|_\infty \leqslant 1$, then there exists a finite Blaschke product u_0 (with $\leqslant n$ zeros) whose power series expansion agrees with that of h in the first n-coefficients.

PROOF OF LEMMA 1. For the given $h \in H^\infty$ and inner function b, let $f = \bar{b}h$, so that $d(f, H^\infty) < 1$. Applying Theorem 1 to f, we get a unimodular function u_0 in $\bar{b}h + H^\infty$ with $d(u_0, H_0^\infty) = 1$. We claim that $\bar{u}_0 \in H^\infty[\bar{b}]$. To see this, let $\varphi = u_0\bar{z}$. Then $d(\varphi, H^\infty) = d(u_0, H_0^\infty) = 1$ and $d(z\varphi, H^\infty) = d(u_0, H^\infty) = d(\bar{b}h, H^\infty) < 1$. So we can apply Theorem 2 to φ, concluding that $\bar{\varphi} \in H^\infty[\varphi] = H^\infty[u_0\bar{z}] = H^\infty[\bar{b}h\bar{z}] \subset H^\infty[\bar{b}]$. So $\bar{u}_0 = \bar{z}\bar{\varphi} \in H^\infty[\bar{b}]$. Let $b_0 = u_0 b$, so that $b_0 \in h + u_0 H^\infty$. Then b_0 is inner and $\bar{b}_0 = \bar{u}_0\bar{b} \in H^\infty[\bar{b}]$.

The following theorem is our main result.

THEOREM 3. *Let B be any closed subalgebra of L^∞ containing H^∞. Let C_B be the C^*-algebra generated by inner functions that are invertible in B. Then the linear span $H^\infty + C_B$ is a closed algebra, and is equal to B.*

To prove Theorem 3, we first make a simple observation which is a direct consequence of Lemma 1 above.

REMARK. If B is a closed subalgebra of L^∞ containing H^∞, if b is an inner function in C_B, and if h is in H^∞, then there exist functions $u \in C_B$ and $g \in H^\infty$ with $\|u\| \leqslant 2\|h\|$, $\|g\| \leqslant 3\|h\|$ and $\bar{b}h = u + g$.

PROOF OF THEOREM 3. Since B is a Douglas algebra, for any $f \in B$, there exists an inner function $b_1 \in C_B$, and an $h_1 \in H^\infty$ with $\|f - \bar{b}_1 h_1\| < \frac{1}{2}\|f\|$. Applying the above remark to the functions h_1 and b_1, we get $u_1 \in C_B$, $g_1 \in H^\infty$, with $\|u_1\| \leqslant 2\|h_1\| \leqslant 4\|f\|$, $\|g_1\| \leqslant 3\|h_1\| \leqslant 6\|f\|$, and $\bar{b}_1 h_1 = u_1 + g_1$. Let $f_1 = f - \bar{b}_1 h_1$; then $f_1 \in B$, and we can apply the above procedure to the function f_1, getting an inner function $b_2 \in C_B$, an H^∞ function h_2, a $u_2 \in C_B$, and a $g_2 \in H^\infty$, with $\|f_1 - \bar{b}_2 h_2\| \leqslant \frac{1}{2}\|f_1\|$, $\bar{b}h_2 = u_2 + g_2$, $\|u_2\| \leqslant 2\|h_2\| \leqslant 4\|f_1\|$, and $\|g_2\| \leqslant 3\|h_2\| \leqslant 6\|f_1\|$. Iterating, we get $f_n \in B$, $u_n \in C_B$, $g_n \in H^\infty (f_0 = f)$ $n = 1, 2, 3, \ldots$ such that

(1) $$f_n = f_{n-1} - (u_n + g_n), \qquad \|f_n\| \leqslant \tfrac{1}{2}\|f_{n-1}\|$$

(2) $$\|u_n\| \leqslant 4\|f_{n-1}\|, \qquad \|g_n\| \leqslant 6\|f_{n-1}\|.$$

Hence $u = \sum_{n=1}^{\infty} u_n$ is in C_B, $g = \sum_{n=1}^{\infty} g_n$ is in H^∞, and $f = \sum_{n=1}^{\infty}(u_n + g_n) = u + g \in H^\infty + C_B$. Since the linear span $H^\infty + C_B$ is obviously contained in B, we have proved the theorem.

For B a closed subalgebra of L^∞ containing H^∞, let $Q_B = B \cap \overline{B}$. Then Q_B is the largest C^*-algebra contained in B. As Q_B contains C_B, the following statement follows immediately from Theorem 3.

COROLLARY 1. *If B is any closed subalgebra of L^∞ containing H^∞, then $B = H^\infty + Q_B$.*

3. **Descriptions of C^*-algebras C_B, Q_B for $B = H^\infty[\bar{b}]$.** Suppose b is an inner function and $B = H^\infty[\bar{b}]$ is the closed subalgebra of L^∞ generated by H^∞ and $\bar{b}$. This section concerns the C^*-algebras C_B and Q_B of B. It turns out, though the general description of C_B and Q_B as subalgebras of L^∞ seem difficult to give, we can give a quite satisfactory description of the subspace of BMO corresponding to C_B and Q_B.

For a function f in L^1 of ∂D and a subarc I of ∂D, let $f_I = (1/|I|)\int_I f(t)\,dt$, where dt denotes the Lebesgue measure on ∂D and, $|I|$ denotes the arc length of the arc I. For each real number a, $0 < a \leqslant 2\pi$, we define $M_a(f) = \sup_{|I| \leqslant a}(1/|I|)\int_I |f - f_I|\,dt$, and we let $\|f\|_* = \lim_{a\to 2\pi} M_a(f)$ and $M_0(f) = \lim_{a\to 0} M_a(f)$. We say that the function f has bounded mean oscillation, or belongs to BMO, if $\|f\|_* < \infty$. We say the function f has vanishing mean oscillation, or belongs to VMO, if $M_0(f) = 0$.

For a subarc I of ∂D with center e^{it} and measure 2δ, let $R(I) = \{re^{i\theta} \mid |\theta - t| \leqslant \delta, 1 - \delta \leqslant r < 1\}$. A finite positive measure μ on D is said to be a Carleson measure if there exists a constant C such that $\mu(R(I)) \leqslant C|I|$ for all subarcs I of ∂D. One of the main results of C. Fefferman and E. M. Stein [22, Theorem 3] is the following theorem, which gives several alternative descriptions of the space BMO: (This is the special case of their theorem for the unit circle.)

THEOREM 4 (FEFFERMAN AND STEIN). *For a function f defined on ∂D the following conditions are equivalent*:

(1) $f \in BMO$.

(2) *$f = u + \tilde{v}$ for some functions $u, v \in L^\infty$, where $\tilde{v}$ denotes the harmonic conjugates of v.*

(3) *$f \in L^1$ and the measure μ on D defined by*

$$d\mu = (1 - r)|\nabla f(re^{i\theta})|^2 r\,dr\,d\theta$$

is a Carleson measure, where

$$|\nabla f(re^{i\theta})|^2 = \left|\frac{\partial f}{\partial r}(re^{i\theta})\right|^2 + \frac{1}{r^2}\left|\frac{\partial f}{\partial \theta}(re^{i\theta})\right|^2.$$

Furthermore, if $C = \sup_{|I| \leqslant 2\pi}(1/|I|)\mu(R(I))$, then there exists an absolute constant A_1 with $C \leqslant A_1 \|f\|_*^2$.

In [11], D. Sarason studied the space *VMO*, and gave several alternative descriptions of the space. One of his results is the following theorem:

THEOREM 5 (SARASON). *For a function f defined on ∂D and in BMO, the following conditions are equivalent*:

(1) $f \in VMO$.

(2) $f = u + \tilde{v}$ *for some functions* $u, v \in C$.

Sarason's work suggests that we look at the subspace $C_B + \tilde{C}_B$ in *BMO*. As we shall see later, this subspace is a closed subspace of *BMO*, and there are several alternative descriptions of the space analogous to those of *BMO* in the Fefferman-Stein theorem.

We begin with some notations and definitions.

For each $0 < \delta < 1$, let G_δ be the region $\{f \in D|\ |b(z)| \geqslant 1 - \delta\}$. For each $\delta > 0$, and real number $a_0 \geqslant 1$, let $\mathcal{I}_{(\delta, a_0)}$ denote the collections of all subarcs I of the circle of the form $I = \{e^{it}|\ |t - \theta| \leqslant a(1 - r)\}$ for some point $re^{i\theta} \in G_\delta$, and some real number $1 \leqslant a \leqslant a_0$ with $a(1 - r) \leqslant \pi$. For each $\varepsilon > 0$, let a_ε be the smallest integer 2^N such that $2\sum_{n=N}^{\infty}(1 + 2n)/2^n \leqslant \varepsilon$. We define VMO_B to be the space of all functions f in *BMO* with the following property: For every $\varepsilon > 0$, there exists a $\delta > 0$, such that $(1/|I|)\int_I |f - f_I| \leqslant \varepsilon$ whenever $I \in \mathcal{I}_{(\delta, a_\varepsilon)}$.

The following theorem is our main result.

THEOREM 6. *For a function f defined on ∂D and in BMO, the following conditions are equivalent*:

(1) $f \in VMO_B$.

(2) $f = u + \tilde{v}$ *for some functions* u, v *in* $Q_B = B \cap \overline{B}$.

(2)(a) $f = u + \tilde{v}$ *for some functions* u, v *in* C_B.

(3) *Given* $\varepsilon > 0$, *there exists some* $\delta > 0$ *such that the measure* μ_δ *on D defined by* $d\mu_\delta = \chi_{G_\delta}(1 - r)|\nabla f|^2 r\, dr\, d\theta$ *is a Carleson measure with* $\mu_\delta(R(I)) \leqslant \varepsilon|I|$ *for all subarcs* I *of* ∂D.

The equivalence between conditions (1) and (2) in the above theorem have been verified for some specific Douglas algebras by S. Axler [12], T. Weight [13] and the author [10]. Condition (3) has been established in the special case $B = H^\infty + C$ by D. Stegenga [23].

Let L_B denote the collection of functions in *BMO* satisfying condition (3) above. To prove the above theorem, we first quote two results from [3].

LEMMA 2 [3, THEOREM 6]. *Suppose $f \in L_B$. There is an absolute constant C such that for every $\varepsilon > 0$,*

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)b^n(t)k(t)\,dt\right| \leqslant C\varepsilon^{1/2}\|f\|_* \|k\|_1$$

for all $k \in H_0^1 \cap H^\infty$, when the positive integer n is sufficiently large.

LEMMA 3 [3, THEOREM 8]. $L_B \cap L^\infty = Q_B\ (= B \cap \overline{B})$.

REMARK. In [3], Lemma 2 was proved for functions $f \in L_B \cap L^\infty$, but the same proof, with the constant $\|f\|$ replaced by $\|f\|_*$ whenever it appears, will work for functions $f \in L_B$.

LEMMA 4 (JOHN-NIRENBERG [24, LEMMA 1′]). *Suppose f is a function in BMO and I is a subarc of ∂D. For each $s > 0$, let $\lambda(s)$ be the set of points on I where $|f - f_I| > s$, and let $|\lambda(s)|$ denote the measure of $\lambda(s)$. Then there exist constants A, α, and s_0 (independent of f) such that*

$$|\lambda(s)| \leqslant \frac{A}{\|f\|_*}\left(\int_I |f - f_I|\,dt\right)\cdot e^{-\alpha s/\|f\|_*}$$

for all $s \geqslant \|f\|_* s_0$.

The estimates in the following lemmas will be used to establish the equivalence of (1) and (2), (3) in Theorem 6. The proofs presented here are only slight modifications of those in [22].

LEMMA 5. *Let $f \in BMO$ and $(1/|I|)\int_I |f - f_I|\,dt \leqslant \varepsilon$. Then*

$$(1/|I|)\int_I |f - f_I|^2\,dt \leqslant \varepsilon C_1 \|f\|_*$$

where C_1 is an absolute constant.

PROOF. The lemma follows easily from Lemma 4 above, and the observation that if we let $t_0 = \|f\|_* s_0$ (s_0 as in Lemma 4), then the condition $(1/|I|)\int_I |f - f_I|\,dt \leqslant \varepsilon$ implies that $|\lambda(t_0)|t_0 \leqslant \varepsilon|I|$.

LEMMA 6. *Let f be a function in VMO_B, and let $\varepsilon > 0$. Let δ be as in the definition of VMO_B. Suppose $z_0 = r_0e^{i\theta_0} \in G_\delta$, $r_0 > \frac{1}{2}$. Let $I_0 = \{e^{it}\,|\,|t - \theta_0| \leqslant 1 - r_0\}$. Then*

$$A(I_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t) - f_{I_0}|P(r_0, \theta_0 - t)\,dt \leqslant C_2\varepsilon,$$

where C_2 is a constant depending only on $\|f\|_*$.

PROOF. Let I_n be the arc with the same center as I_0 and measure $2^n|I_0|$. Let N be the integer such that $a_\varepsilon = 2^N$. We carry out the proof for the case

$2^N|I_0| < \pi$. (The same proof works in the contrary case, with a slight change in the constant C_2.) We have

$$A(I_0) = \frac{1}{2\pi}\sum_{n=0}^{N}\int_{I_n\setminus I_{n-1}} |f(t) - f_{I_0}|P(r_0,\theta_0 - t)\,dt$$
$$+\frac{1}{2\pi}\int_{\partial D\setminus I_N} |f(t) - f_{I_0}|P(r_0,\theta_0 - t)\,dt.$$

The estimate

$$|f_{I_{n-1}} - f_{I_n}| \leqslant \frac{1}{|I_{n-1}|}\int_{I_{n-1}} |f - f_{I_n}|\,dt \leqslant \frac{|I_n|}{|I_{n-1}}\frac{1}{|I_n|}\int_{I_n} |f - f_{I_n}|\,dt \leqslant 2\varepsilon,$$

valid for $n = 0, 1, \dots, N$ gives

$$|f_{I_0} - f_{I_n}| \leqslant \sum_{k=1}^{n} |f_{I_{k-1}} - f_{I_k}| \leqslant 2n\varepsilon, \qquad n = 0, 1, 2, \dots, N,$$

which, together with an elementary estimate of $P(r_0,\theta_0 - t)$, yields

$$\int_{t\in I_n - I_{n-1}} |f(t) - f_{I_0}|\,|P(r_0,\theta_0 - t)|\,dt$$
$$\leqslant \frac{\pi^2}{|I_0|}\frac{1}{2^{2n-1}}\left(\int_{t\in I_n} (|f(t) - f_{I_n}| + |f_{I_n} - f_{I_0}|)\,dt\right)$$
$$= \frac{\pi^2}{2^{n-1}}\frac{1}{|I_n|}\int_{t\in I_n} (|f(t) - f_{I_n}| + 2n\varepsilon)\,dt$$
$$\leqslant \pi^2\frac{1 + 2n}{2^{n-1}}\varepsilon, \qquad n = 0, 1, \dots, N.$$

Hence

$$\sum_{n=0}^{N}\frac{1}{2\pi}\int_{t\in I_n\setminus I_{n-1}} |f(t) - f_{I_0}|P(r_0,\theta_0 - t) \leqslant \left(\sum_{n=0}^{N}\frac{1 + 2n}{2^n}\right)\varepsilon\pi. \tag{3}$$

Using similar estimates we get, for N_1 the largest integer such that $2^{N_1}|I| < 2\pi$,

$$\frac{1}{2\pi}\int_{\partial D\setminus I_N} |f(t) - f_{I_0}|P(r_0,\theta_0 - t)\,dt$$
$$= \frac{1}{2\pi}\sum_{n=N}^{N_1}\int_{I_{n+1}\setminus I_n} |f(t) - f_{I_0}|P(r_0,\theta_0 - t)\,dt$$
$$+\frac{1}{2\pi}\int_{\partial D\setminus I_{N_1}} |f(t) - f_{I_0}|P(r_0,\theta_0 - t)\,dt \tag{4}$$
$$\leqslant \pi\sum_{n=N}^{\infty}\frac{1 + 2n}{2^n}\|f\|_* + \pi\frac{1 + 2^N}{2^N}\|f\|_*$$
$$\leqslant \varepsilon\pi\|f\|_* \quad \text{(by definition of } a_\varepsilon\text{)}.$$

Hence $A(I_0) \leqslant C_2\varepsilon$ for $C_2 = (\sum_{n=0}^{\infty}(1 + 2n)/2^n + \|f\|_*)\pi$.

PROOF OF THEOREM 6. We will first establish the equivalence between (2)(a) and (3).

(2)(a) $\Rightarrow$ (3). Suppose $f = u + \tilde{v}$ for some $u, v \in C_B$. Since $C_B \subset Q_B$, by Lemma 3, $u \in L_B$. Since the space L_B is closed under harmonic conjugation, the same reasoning gives that $\tilde{v} \in L_B$. Hence f is in L_B.

(3) $\Rightarrow$ (2)(a). Suppose f is in L_B. Since L_B is closed under complex conjugation, we may assume without loss of generality that f is real-valued. By Theorem 4, there exist u, v in L^∞ such that $f = u + \tilde{v}$. Let $g = u + iv$. Then $g \in L^\infty$. We have $\int_{-\pi}^{\pi}(f - g)h\,dt = \int_{-\pi}^{\pi}(\tilde{v} - iv)h\,dt = 0$ for all $h \in H^1 \cap H^\infty$. Applying Lemma 2 to f, we get, for any ε, $|(1/2\pi)\int_{-\pi}^{\pi} gb^n k\,dt| < C\varepsilon^{1/2}\|k\|_1\|f\|_*$, for all $k \in H_0^1 \cap H^\infty$ and for n sufficiently large. Since the quotient space L^∞/H^∞ is the dual of the space H_0^1, this implies that $d(gb^n, H^\infty)$ (which equals the norm of the functional that gb^n induces on H_0^1) tends to 0 when n tends to ∞. Hence $g \in B$. Applying Theorem 3, we can write $g = r + h$ for some $r \in C_B$, $h \in H^\infty$. Thus $f - i\tilde{f} = g - i\tilde{g} = r - i\tilde{r} \in C_B + \tilde{C}_B$. Taking complex conjugates on both sides of the above equality, we get $f + i\tilde{f} = \bar{r} + i\tilde{\bar{r}} \in C_B + \tilde{C}_B$. Hence $f \in C_B + \tilde{C}_B$. The proof is complete.

It now follows easily that (2) and (3) are equivalent, since (2) $\Rightarrow$ (3) by the same reasoning as (2)(a) $\Rightarrow$ (3), and (3) $\Rightarrow$ (2)(a) $\Rightarrow$ (2).

The equivalence between (1) and (2), (3) follows from the following three assertions:

(a) $B \cap \overline{B} \subset VMO_B$.

(b) If $u \in VMO_B \cap L^\infty$ then $\tilde{u} \in VMO_B$.

(c) $VMO_B \subset L_B$.

PROOF OF (a). Since $B \cap \overline{B}$ is a C^*-algebra, and so is spanned by its unimodular functions, and since $VMO_B \cap L^\infty$ is a closed subspace of L^∞, it suffices to show every unimodular function in $B \cap \overline{B}$ is in VMO_B.

For fixed $\varepsilon > 0$, and f a unimodular function invertible in $B \cap \overline{B}$, let ε_1 be any positive number less than $[2(4\pi(1 + a_\varepsilon^2) + 1)]^{-1}\varepsilon$, and choose $\delta > 0$ so that $|f(z)| \geqslant 1 - \varepsilon_1^3$ for all points z in the region G_δ. (The existence of such δ is asserted in the proof of Theorem 7 in [3].) Then apply the same type of estimate as in the proof of Theorem 3 in [11], we can show that whenever I is a subarc of the form $\{e^{it} \mid |t - \theta| \leqslant a(1 - r)\}$ in $\mathcal{G}(\delta, a_\varepsilon)$, we have

$$\frac{1}{|I|}\int_I |f(e^{it}) - f_I|\,dt \leqslant 2\left(\frac{4\pi(1 + a^2)}{a} + 1\right)\varepsilon_1$$

$$\leqslant 2(4\pi(1 + a_\varepsilon^2) + 1)\varepsilon_1 < \varepsilon.$$

Hence f is in VMO_B.

PROOF OF (b). Let u be a function in VMO_B, and assume without loss of generality that u is real valued. Given $\varepsilon > 0$, let $\varepsilon_1, \varepsilon_2$ be two numbers satisfying $2a_\varepsilon^2 < 2a_{\varepsilon_1}^2 < a_{\varepsilon_2}$. Choose $\delta > 0$ corresponding to ε_2 for u as in the definition of VMO_B, i.e.,

$$\sup_{I \in \mathcal{G}_{(\delta, a_{\varepsilon_2})}} \frac{1}{|I|} \int_I |u - u_I| \, dt \leqslant \varepsilon_2 .$$

Fix a subarc $I \in \mathcal{G}_{(\delta, a_\varepsilon)}$, we will show that $(1/|I|) \int_I (\tilde{u} - \tilde{u}_I) \, dt \leqslant \varepsilon$. Let J be the subarc with the same center as I and twice the dimensions. Let $u_1 = \chi_J(u - u_J)$, $u_2 = \chi_{\partial D \setminus J}(u - u_J)$; then we have

$$\frac{1}{|I|} \int_I |\tilde{u} - \tilde{u}_I| \, dt \leqslant \sum_{j=1}^{2} \frac{1}{|I|} \int_I |\tilde{u}_j - (\tilde{u}_j)_I| \, dt. \tag{5}$$

Since $\|\tilde{u}_1\|_2 \leqslant \|u_1\|_2$,

$$\frac{1}{|I|} \int_I |\tilde{u}_1|^2 \, dt \leqslant \frac{1}{|I|} \int_{\pi}^{\pi} |\tilde{u}_1|^2 \, dt \leqslant \frac{1}{|I|} \int_{-\pi}^{\pi} |u_1|^2 \, dt$$
$$= \frac{1}{|I|} \int_J |u - u_J|^2 \, dt \leqslant 2\varepsilon_2 C_1 \|u\|_*.$$

The last step of the above inequality follows since $J \in \mathcal{G}_{(\delta, a_{\varepsilon_2})}$, so we can apply Lemma 5. Thus

$$\frac{1}{|I|} \int_I |\tilde{u}_1 - (\tilde{u}_1)_I|^2 \, dt \leqslant 2\left(\frac{1}{|I|} \int_I |\tilde{u}_1|^2 \, dt + |(u_1)_I|^2 \right)$$
$$\leqslant 2 \cdot 4\varepsilon_2 C_1 \|u\|_* = 8\varepsilon_2 C_1 \|u\|_*,$$

so

$$\frac{1}{|I|} \int_I |u_1 - (u_1)_I| \, dt \leqslant (8\varepsilon_2 C_1 \|u\|)^{1/2}. \tag{6}$$

To estimate $(1/|I|) \int_I |\tilde{u}_2 - (\tilde{u}_2)_I| \, dt$, we let $f = u_2 + i\tilde{u}_2$, i.e.

$$f(z) = \frac{1}{2\pi} \int_{\partial D \setminus J} \frac{e^{it} + z}{e^{it} - z} (u - u_J) \, dt.$$

This formula holds on I as well as in D. Differentiating f, we get

$$f'(z) = \frac{1}{\pi} \int_{\partial D \setminus J} \frac{e^{it}}{(e^{it} - z)^2} (u(t) - u_J) \, dt.$$

Let $e^{i\theta_0}$ be the center of I, and $z_0 = (1 - \frac{1}{2}|I|)e^{i\theta_0}$; then for $e^{is} \in I$, we have

$$\begin{aligned}
|f'(e^{is})| &\leq \frac{1}{\pi}\int_{\partial D\setminus J}\frac{|u(t)-u_J|}{|e^{it}-e^{is}|^2}\,dt \\
&\leq \frac{C_3}{\pi}\int_{\partial D\setminus J}\frac{|u(t)-u_J|}{|e^{it}-z_0|^2}\,dt \quad \text{where } C_3 = \left(1+\frac{\pi^2}{2}\right)^2 \\
&\leq \frac{C_3}{\pi}\left(\int_{\partial D\setminus J}\frac{|u(t)-u_I|}{|e^{it}-z_0|^2}\,dt + \frac{2}{|I|}|u_I-u_J|\right) \\
&\leq \frac{C_3}{\pi}\left(\int_{-\pi}^{\pi}\frac{|u-u_I|}{|e^{it}-z_0|^2}\,dt + \varepsilon_2\frac{1}{|I|}\right).
\end{aligned}$$

Since each interval with the same center as I and with length $\leq a_{\varepsilon_1}|I|$ is in a_{ε_2}; we can apply the same method as in Lemma 6 to estimate $S_1 = (1/2\pi)\int_{-\pi}^{\pi}|u-u_1|/|e^{it}-z_0|^2\,dt$ and get

$$S_1 \leq (2/|I|)(C_2\varepsilon_2 + \varepsilon_1\pi\|u\|_*).$$

Hence

$$\begin{aligned}
|f'(e^{is})| &\leq \frac{1}{|I|}\frac{2C_3}{\pi}(C_2\varepsilon_2 + \varepsilon_1\pi\|u\|_* + 4\varepsilon_2) \\
&= \frac{1}{|I|}C_4 \quad \text{for some constant } C_4 (\text{depends on } \varepsilon_1, \varepsilon_2).
\end{aligned}$$

Hence the oscillation of f over I does not exceed C_4. And the same applies to $Imf = \tilde{u}_2$, i.e.

$$\frac{1}{|I|}\int_I |\tilde{u}_2 - (\tilde{u}_2)_I|\,dt \leq C_4. \tag{7}$$

From (5), (6), (7) we get $(1/|I|)\int_I |u-u|_I\,dt \leq \varepsilon$ if ε_1, ε_2 are chosen sufficiently small. This concludes the proof of (b).

PROOF OF (c). The proof given here is a slight modification of the proof of Lemma 4 in [22]. Suppose f is a function in VMO_B. Given $\varepsilon > 0$, choose ε_1 smaller then ε, to be fixed later, and let δ correspond to ε_1 for f as in the definition of VMO_B. Let $z_0 = r_0 e^{i\theta_0} \in G_\delta$ with $r_0 \geq \frac{1}{2}$, and let $S(\theta_0, r_0)$ be the region $\{re^{i\theta} \mid |\theta-\theta_0| \leq 4(1-r_0), r_0 \leq r < 1\}$. Then as indicated in Theorem 6 in [3], to show f is in L_B, it suffices to show

$$\iint_{S(\theta_0,r_0)} (1-r)|\nabla f|^2 r\,dr\,d\theta \leq \varepsilon(1-r_0).$$

To do this, let $J = \{e^{it} \mid |t-\theta_0| \leq 5(1-r_0)\}$, $I = \{e^{it} \mid |t-\theta_0| \leq 1-r_0\}$, $f_1 = \chi_J(f-f_J)$, $f_2 = \chi_{\partial D - J}(f-f_J)$. Then

$$\iint_{S(\theta_0,r_0)} (1-r)|\nabla f_1|^2 r\,dr\,d\theta \leqslant \iint_D (1-r)|\nabla f_1|^2 r\,dr\,d\theta$$

$$\leqslant \iint_D |\nabla f_1|^2 r \log\frac{1}{r}\,dr\,d\theta = \frac{1}{2}\int_J |f-f_J|^2\,dt \leqslant \frac{|J|}{2}\varepsilon_1 C_1 \|f\|_*.$$

The last inequality follows if we assume $a_{\varepsilon_1} > 5$ (which is so when ε_1 is small), so that $J \in \mathcal{G}_{(\delta,a_{\varepsilon_1})}$, enabling us to apply Lemma 5. Moreover,

$$|\nabla f_2(re^{i\theta})| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |\nabla P(r,\theta-t)|\,|f_2(e^{it})|\,dt$$

$$= \frac{1}{\pi}\int_{\partial D\setminus J} \frac{1}{|e^{it}-re^{i\theta}|^2}|f(e^{it})-f_J|\,dt.$$

Hence if $re^{i\theta} \in S(\theta_0, r_0)$, we have

$$|\nabla f_2(re^{i\theta})| \leqslant \frac{1}{\pi}C_5\int_{-\pi}^{\pi}\frac{|f(e^{it})-f_J|}{|e^{it}-z_0|^2}dt \quad \left(C_5 = 2\left(1+\frac{17\pi^2}{2}\right)\right)$$

$$\leqslant \frac{1}{\pi}C_5\int_{-\pi}^{\pi}\frac{|f(e^{it})-f_I|}{|e^{it}-z_0|^2}dt + \frac{2\pi}{1-r_0^2}|f_I-f_J|$$

$$\leqslant \frac{4}{|I|}C_5(2\pi C_2\varepsilon_1 + 5\varepsilon_1)\,(\text{by Lemma 6})$$

$$= C_6\frac{1}{|I|}\varepsilon_1 \quad \text{for some constant } C_6.$$

Thus

$$\iint_{S(\theta_0,r_0)} (1-r)|\nabla f_2|^2 r\,dr\,d\theta \leqslant \iint_{S(\theta_0,r_0)} C_6^2\varepsilon_1^2\frac{1}{|I|^2}(1-r)\,dr\,d\theta \tag{9}$$

$$= \tfrac{1}{2}C_6^2\varepsilon_1^2(1-r_0).$$

Since $|\nabla f|^2 \leqslant 2(|\nabla f_1|^2 + |\nabla f_2|^2)$, from inequalities (8) and (9), we get the desired conclusion $\iint_{S(\theta_0,r_0)}(1-r)|\nabla f|^2 r\,dr\,d\theta \leqslant \varepsilon(1-r_0)$ if we choose a suitable ε_1. Thus the proof of the theorem is complete.

4. Description of C^*-algebras C_B, Q_B for $H^\infty \subset B \subset L^\infty$. Suppose B is a closed subalgebra of L^∞ which contains H^∞. Then, as stated in §1, B is a Douglas algebra. Suppose B is generated by H^∞ and a collection $\{b_\lambda\}$ of conjugates of inner functions, where λ runs over some index set E. For each $\lambda \in E$, let B_λ be the closed subalgebra $H^\infty[\bar{b}_\lambda]$. For each finite subset $F \subset E$, let B_F be the closed algebra generated by B_λ, for all $\lambda \in F$, and let b_F be the inner function $\prod_{\lambda\in F} b_\lambda$. Then it is easy to see that $B_F = H^\infty[\bar{b}_F]$. For each $\delta > 0$, let $G_\delta(b_F) = \{z \in D \mid |b_F(z)| \geqslant 1-\delta\}$. For each finite subset $F \subset E$, each $\delta > 0$, and each real number a_0, let $\mathcal{I}_{(\delta,a_0)}(F) = \{I \mid I = \{e^{it} \mid |t - \theta| \leq a(1 - r)$, with $re^{i\theta} \in G_\delta(b_F)$, $a(1 - r) \leq \pi\}$, where $1 \leq a \leq a_0\}$. For each $\varepsilon > 0$, let a_ε be the same integer as in §3. We define VMO_B to be the space of all functions f in BMO with the following properties:

For every $\varepsilon > 0$, there exists some $\delta > 0$, and some finite subset F of E, such that $(1/|I|)\int_I |f - f_I|\,dt \leq \varepsilon$ whenever $I \in \mathcal{I}_{(\delta,a_\varepsilon)}(F)$. The following theorem is parellel to Theorem 6:

THEOREM 7. *Let B, b_λ, E, F, δ, b_F, $G_\delta(b_F)$, VMO_B be defined as above. For a function f defined on ∂D and in BMO, the following conditions are equivalent*:

(1) $f \in VMO_B$.

(2) $f = u + \tilde{v}$ *for some functions u, v in* $Q_B = B \cap \overline{B}$.

(2)(a) $f = u + \tilde{v}$ *for some functions u, v in* C_B.

(3) *Given $\varepsilon > 0$, there exists some $\delta > 0$ and some finite subset F of E, such that the measure $\mu_\delta(F)$ on D defined by $d\mu_\delta(F) = \chi_{G_\delta(b_F)}(1 - r)|\nabla f|^2 r\,dr\,d\theta$ is a Carleson measure, and $\mu_\delta(F)(R(I)) \leq \varepsilon|I|$ for all subarcs I of ∂D.*

In view of Theorem 4 in [3] (which says that the collection of functions in L^∞ which satisfy condition (3) in Theorem 7 above is the C^*-algebra Q_B) and the argument used to establish Theorem 7 in [3], it is easy to see the same proof of Theorem 6 which pertains to the special case $B = H^\infty[\bar{b}]$, also work for general algebra B after some slight modification. We will skip the details here.

The following corollary is an immediate consequence of Lemma 3 (which holds also for general B) and Theorem 7. It gives some information about the boundary behavior of functions in the subalgebra Q_B, and hence answers another question proposed by D. Sarason in [5].

COROLLARY. $Q_B = VMO_B \cap L^\infty = (C_B + \tilde{C}_B) \cap L^\infty$.

REFERENCES

1. R. G. Douglas, *On the spectrum of Toeplitz and Wiener-Hopf operators*, Abstract Spaces and Approximation (Proc. Conf., Oberwolfach, 1968), Birkhäuser, Basel, 1969, pp. 53–66. MR **41** #4274.

2. D. E. Marshall, *Subalgebras of L^∞ containing H^∞*, Acta Math. **137** (1976), 91–98.

3. S. Y. Chang, *A characterization of Douglas subalgebras* Acta Math. **137** (1976), 81–89.

4. D. E. Sarason, *Algebras of functions on the unit circle*, Bull. Amer. Math. Soc. **79** (1973), 286–299. MR **48** #2777.

5. D. E. Sarason, *Algebras between L^∞ and H^∞*, Lecture Notes in Math., vol. 512, Springer-Verlag, Berlin and New York, 1976, pp. 117–129.

6. R. G. Douglas and W. Rudin, *Approximation by inner functions*, Pacific J. Math. **31** (1969), 313–320. MR **40** #7814.

7. K. Hoffman and I. M. Singer, *Maximal subalgebras of continuous functions*, Acta Math. **103** (1960), 217–241. MR **22** #8318.

8. H. Helson and D. E. Sarason, *Past and future*, Math. Scand. **21** (1967), 5–16 (1968), MR **38** #5282.

9. A. M. Davie, T. W. Gamelin and J. Garnett, *Distance estimates and pointwise bounded density*, Trans. Amer. Math. Soc. **175** (1973), 37–68. MR **47** #2068.

10. S. Y. Chang, *On the structure and characterization of some Douglas algebras*, Amer. J. Math. (to appear).

11. D. E. Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. **207** (1975), 391–405.

12. S. Axler, *Some properties of* $H^\infty + L_E$ (preprint).

13. T. Weight, Dissertation, University of California, Los Angeles, 1976.

14. A. Bernard, J. B. Garnett and D. E. Marshall, *The algebra generated by inner functions* (preprint).

15. V. M. Adamjan, D. Z. Arov, and M. G. Kreĭn, *Infinite Hankel matrices and generalized Carathéodory-Fejér and I. Schur problems*, Funkcional. Anal. i Priložen **2** (1968), 1–17 = Functional Anal. Appl. **2** (1968), 269–281.

16. M. Lee and D. E. Sarason, *The spectra of some Toeplitz operators*, J. Math. Anal. Appl. **33** (1971), 529–543. MR **43** #960.

17. R. G. Douglas and D. E. Sarason, *Fredholm Toeplitz operators*, Proc. Amer. Math. Soc. **26** (1970), 117–120. MR **41** #4275.

18. L. A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13** (1966), 285–288. MR **34** #1846.

19. R. G. Douglas, *Banach algebra technique in operator theory*, Academic Press, New York, 1972.

20. R. Nevanlinna, *Über beschränkte analytisch Funktionen*, Ann. Acad. Sci. Fenn. Ser. A **32** (1929), No. 7.

21. C. Carathéodory and L. Fejér, *Über den Zusammenhang der Extremen von harmonischen Funktimen mit ihren Koeffizientan und über den Picard-Landan'schen Satz*, Rend. Circ. Mat. Palermo **32** (1911), 218–239.

22. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

23. D. A. Stegenga, *Bounded Toeplitz operators on H^1 and applications of the duality between H^1 and the functions of bounded mean oscillation*, Amer. J. Math. **98** (1976), 573–589.

24. F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426. MR **24** #A1348.

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BUFFALO, BUFFALO, NEW YORK 14226

Current address: School of Mathematics, Institute for Advance Study, Princeton, New Jersey 08540